T-L Plane-Based Real-Time Scheduling for Homogeneous Multiprocessors

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Abstract

We consider optimal real-time scheduling of periodic tasks on multiprocessors—i.e., satisfying all task deadlines, when the total utilization demand does not exceed the utilization capacity of the processors. We introduce a novel abstraction for reasoning about task execution behavior on multiprocessors, called *T-L plane* and present *T-L plane-based real-time scheduling* algorithms. We show that scheduling for multiprocessors can be viewed as scheduling on repeatedly occurring T-L planes, and feasibly scheduling on a single T-L plane results in an optimal schedule. Within a single T-L plane, we analytically show a sufficient condition to provide a feasible schedule. Based on these, we provide two examples of T-L plane-based real-time scheduling algorithms, including non-work-conserving and work-conserving approaches. Further, we establish that the algorithms have bounded overhead. Our simulation results validate our analysis of the algorithm overhead. In addition, we experimentally show that our approaches have a reduced number of task migrations among processors, compared to a previous algorithm.

Key words: real-time scheduling, optimality, multiprocessor systems

1 Introduction

Multiprocessor architectures (e.g., Symmetric Multi-Processors or SMPs, Single Chip Heterogeneous Multiprocessors or SCHMs) are becoming more attractive for embedded systems, primarily because major processor manufacturers (Intel, AMD) are rapidly decreasing the prices and increasing the cost/performance.

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Responding to this trend, real-time operating system (RTOS) vendors are increasingly providing multiprocessor platform support. But this exposes the critical need for real-time scheduling for multiprocessors—a comparatively undeveloped area of real-time scheduling which has recently received significant research attention, but is not yet well supported by the RTOS products. Consequently, the impact of cost-effective multiprocessor platforms remains nascent.

Liu [2] first addressed the *multiple-resource scheduling problem* that considers allocating several identical resources (e.g., multiple processors) to a number of periodic tasks, where a task is characterized by two parameters, *execution time* and *period*. A valid schedule must satisfy two constraints: (1) at any instant, at most one task can be executed on any single processor; and (2) no single task can be executed on more than one processor at the same time instant. Under these constraints, a feasible real-time schedule allocates the exact time units of the resource that each task requires for its execution to the task during its period.

One unique aspect of multiprocessor scheduling is the degree of run-time migration that is allowed for job instances of a task across processors (at scheduling events). Example migration models include: (1) *full migration*, where jobs are allowed to arbitrarily migrate among processors during their execution. This usually implies a global scheduling strategy, where a single shared scheduling queue is maintained for all processors, and a system-wide scheduling decision is made by a single (global) scheduling algorithm; (2) *no migration*, where tasks are statically (off-line) partitioned and allocated to processors. At run-time, job instances of tasks are scheduled on their respective processors by processors' local scheduling algorithm, like single processor scheduling; and (3) *restricted migration*, where some form of migration is allowed—e.g., at job boundaries.

Carpenter *et al.* [3] have catalogued multiprocessor real-time scheduling algorithms considering the degree of job migration and the complexity of priority mechanisms employed. The latter includes classes such as (1) *static*, where task priorities never change, e.g., rate-monotonic (RM); (2) *dynamic but fixed within a job*, where job priorities are fixed, e.g., earliest-deadline-first (EDF) [1]; and (3) *fully-dynamic*, where job priorities are dynamic.

The *Proportionate fair (Pfair)* algorithms [4] that allow full migration and fully dynamic priorities have been shown to be theoretically optimal—i.e., they achieve a schedulable utilization bound (below which all tasks meet their deadlines) that equals the total capacity of all processors. Fairness allows

[1] Periodic tasks consist of an infinite sequence of identical activities, jobs, that are regularly activated at a constant rate. Using EDF, once a job starts, its priority does not change until its completion.

all tasks to receive a share of the processor time and simultaneously make progress. However, Pfair algorithms incur significant run-time overhead due to their quantum-based scheduling approach [5,6]—under Pfair, tasks can be decomposed into several small uniform segments, which are then scheduled, causing frequent scheduling and migration.

Zhu *et al.* proposed *boundary fair (BF)* scheduling, which makes scheduling decisions only at period boundaries to reduce the number of scheduling points [7]. It is not as fair as Pfair (fair at any time quantum), but fair enough (only at period boundaries) to get a feasible schedule. Especially when the number of tasks is small (less than 100 in their experiments), they showed that the overhead of BF is less than that of PD (an efficient Pfair algorithm). Note that as the number of tasks increases, the frequency of boundaries also increases and consequently, the number of scheduling points of BF becomes similar to that of PD.

In this paper, we focus on the multiple-resource scheduling problem. We introduce an abstraction for reasoning about the execution behavior of a class of periodic tasks on multiprocessors, *time and local remaining execution-time plane* (abbreviated as the *T-L plane*). The T-L plane makes it possible to envision the entire scheduling activity over time as scheduling in repeated T-L planes of various sizes, so that feasibly scheduling on a single T-L plane results in an optimal schedule for all T-L planes across time.

Moreover, the T-L plane provides a visual model of task execution behavior on multiprocessors, which allows insightful and analytical understanding. Based on this in-depth understanding from the visual model, we present the minimum but sufficient guidelines for designing feasible scheduling algorithms for our task model on multiprocessors. This provides flexibility in the sense that task priorities can be assigned as desired within the constraints of the sufficiency guidelines.

More concretely, T-L plane-based scheduling consists of two phases: the *local parameter decision phase (LD-P)* to establish a T-L plane over a period of time, and the *local scheduling phase (LS-P)* to locally schedule tasks within the established T-L plane. First focusing on LS-P, we analytically derive a sufficient condition to provide a feasible schedule within a single T-L plane. Based on this, we provide two examples of T-L plane-based real-time scheduling algorithms, one non-work-conserving and one work-conserving. The work-conserving approach is for reducing task response times, while the non-work-conserving approach smoothes out the task completion flow. In addition, we experimentally show that our approach outperforms an existing algorithm (McNaughton's [14] which Zhu *et al.* used in [7]) for local scheduling by reducing unnecessary migration.

Thus, the paper's contributions include the T-L plane scheduling abstraction for optimal multiprocessor real-time scheduling of a class of periodic tasks. We also provide a set of sufficient conditions for feasible scheduling of that class. Finally, we provide two T-L plane-based real-time scheduling algorithms—one work-conserving and one non-work-conserving. These have bounded scheduling overhead and simultaneously, lower number of task migrations compared to the existing algorithm mentioned above.

The rest of the paper is organized as follows: In Section 2, we discuss the rationale behind the T-L plane. In Section 3, we analytically discuss the properties of T-L plane-based real-time scheduling. In Section 4, we establish two examples of T-L plane-based scheduling algorithms and establish the upper-bound of the algorithms' overhead. Several experimental results are discussed in Section 5. The paper concludes in Section 6.

2 Preliminaries

2.1 Model

We consider global scheduling, where task migration is not restricted, on an SMP system with M identical processors. The application is assumed to consist of a set of tasks, denoted $\mathbf{T}=\{T_1, T_2,..., T_N\}$. Tasks are assumed to arrive periodically at their release times a_i. Each task T_i has an execution time c_i, and a relative deadline d_i which is the same as its period p_i. The utilization u_i of a task T_i is defined as c_i/d_i and is assumed to be less than 1. Similar to [5,8], we assume that tasks may be preempted at any time, and are independent—i.e., they do not share resources or have any precedences.

The cost of context switches and task migrations are assumed to be negligible, as in [5,8].

2.2 Time and Local Execution Time Plane

In the *fluid* scheduling model, each task executes at a constant rate, which is similar to its utilization demand, at all times [13]. The quantum-based Pfair scheduling algorithm is based on the fluid scheduling model, as the algorithm constantly uses task utilization to track the allocated task execution rate. The Pfair algorithm's success in constructing optimal multiprocessor schedules (meeting all deadlines) for its intended class of tasks can be attributed to *fairness*—informally, all tasks receive a share of the processor time, and thus are able to simultaneously make progress. P-fairness is a strong notion of fairness, which ensures that at any instant, no application is one or more quanta away from its due share (or fluid schedule) [4,10]. The significance of the fairness concept in Pfair's optimality is also supported by the fact that task *urgency*, as represented by the task deadline, is not sufficient for constructing optimal schedules with respect to meeting all deadlines, as we observe from the poor deadline satisfaction ratio [5,11] of global EDF for multiprocessors.

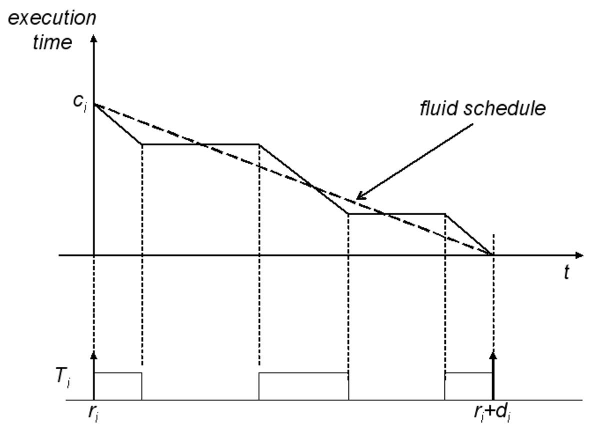


Fig. 1. Fluid Schedule versus a Practical Schedule

Thus, to design an optimal multiprocessor scheduling algorithm for the task model described above, we focus on the fluid scheduling model and the fairness notion. To better understand the multiprocessor real-time scheduling problem we are considering, we create an abstraction, called *T-L plane*, on which tokens representing tasks move over time. (The horizontal location of the token represents the current time and the vertical location of the token represents its remaining execution time. We will describe it in more detail later.) The T-L plane is inspired by the L-C plane abstraction introduced by Dertouzos *et al.* in [12]. We use the T-L plane to depict fluid schedules, and present a new scheduling algorithm that is able to approximate the fluid schedule without using time quanta.

Figure 1 illustrates the fundamental idea behind the T-L plane. For a task T_i with a_i, c_i, and d_i, the figure shows a 2-dimensional plane with time represented on the x-axis and the task's remaining execution time represented on the y-axis. If a_i is assumed as the origin, the dotted line from $(0, c_i)$ to $(d_i, 0)$ depicts the fluid schedule, the slope of which is $-u_i$. Since the fluid schedule is ideal but practically impossible, the fairness of a scheduling algorithm depends on how much the algorithm approximates the fluid schedule path.

When T_i runs like in Figure 1, for example, its execution can be represented as a broken line between $(0, c_i)$ and $(d_i, 0)$. Note that task execution is represented as a line whose slope is -1 since x and y axes are in the same scale, and the non-execution over time is represented as a line whose slope is zero. It is clear that the Pfair algorithm can also be represented on the T-L plane as a broken line based on time quanta.

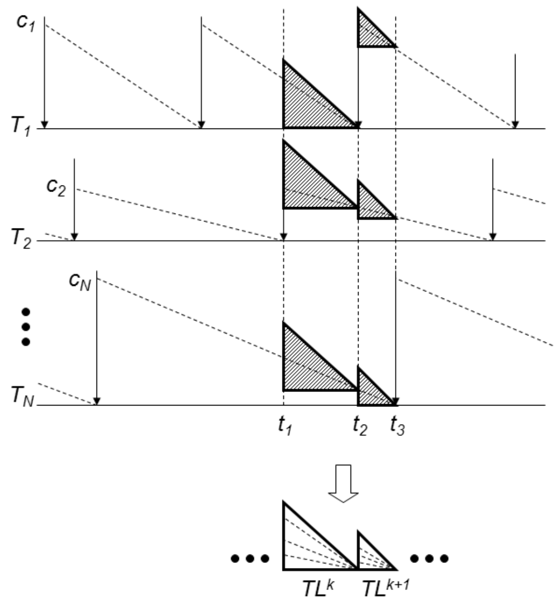


Fig. 2. T-L Planes

When N number of tasks are considered, their fluid schedule can be constructed as shown in Figure 2. After constructing the fluid schedule of each task over time, two consecutive scheduling events are considered, t_1 and t_2, for example.[2] Then, a horizontal line for each task is established from the fluid schedule at the following scheduling event t_2 to the previous scheduling event t_1—more accurately, the vertical location of the horizontal line is determined from the vertical location of the fluid schedule line at the event t_2. The length of the horizontal line is $t_2 - t_1$, which is, not surprisingly, the same as those of all other tasks between t_1 and t_2. At the previous scheduling event t_1, a vertical line of the same length as the horizontal line is drawn for each task, and together with the horizontal line, this yields a right isosceles triangle between t_1 and t_2 for each task. Since the triangles between two consecutive scheduling events for each task are the same size, those N triangles can be overlapped with the fluid schedule of each task inside. The whole schedule over time can be represented as the sequence of these overlapped triangles, $\{TL^0,...,TL^k,...\}$, where k is simply increasing over time. We call this triangle the T-L plane. The size of the k^{th} T-L plane, TL^k, may change over k. The bottom side of each triangle represents time. The left vertical side of each triangle represents the task's remaining execution time, which we call the *local remaining execution time.*

Definition 1 (Local Remaining Execution Time) $l_i^k(t)$ *denotes the* i^{th} *task's*

[2] In this paper, we assume that all tasks have deadlines equal to their periods. Thus, the events occur at task period boundaries.

local remaining execution time at time t on the k^{th} T-L plane, which is supposed to be consumed before the end of the k^{th} T-L plane.

T_i's deadline may not coincide with the end of TL^k, as more than one T-L plane often exists between a task's release and completion. Fluid schedules for tasks are constructed to be overlapped in each TL^k plane, which keeps their slopes the same.

The abstraction of T-L planes is significantly meaningful in scheduling our task model for multiprocessors, because T-L planes are repeated over time, and a feasible scheduling algorithm for a single T-L plane leads to feasible scheduling of all tasks on all repeated T-L planes.

2.3 Scheduling on T-L planes

Figure 3 details a single T-L plane. (When k is omitted, it implicitly means the current k^{th} T-L plane, TL^k. We explicitly use the index k when necessary, for example, when comparing two adjacent T-L planes.)

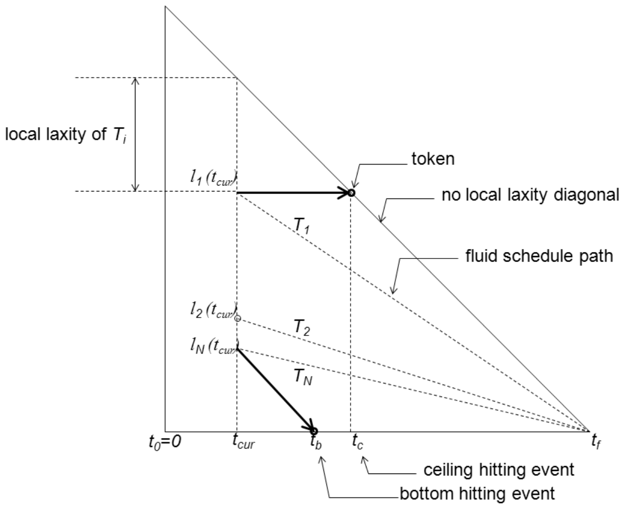


Fig. 3. A Single T-L Plane

The status of each task is represented as a *token* on the T-L plane. The token's location describes the current time as a value on the horizontal axis and the task's remaining execution time as a value on the vertical axis. The remaining execution time of a task here means time that must be consumed before the finishing time of the T-L plane, t_f, and not the task's deadline. Hence, we call it the *local* remaining execution time.

As scheduling decisions are made over time, each task's token moves on the T-L plane. Although the ideal paths of tokens are shown as dotted lines in Figure 3, the tokens are allowed to move only on two paths. (Therefore, tokens can deviate from the ideal paths.) When the task is selected and executed, the token moves diagonally down, as T_N moves. Otherwise, it moves horizontally, as T_1 moves. If M processors are being scheduled and executing, at most M tokens can diagonally move together. The scheduling objective on the T-L plane is to make all tokens arrive at the bottom of the T-L plane before t_f—i.e., to make the local remaining execution times of all tasks be consumed before t_f. We call this successful arrival *locally feasible*. If all tokens are locally feasible on each T-L plane, it is possible for them to be scheduled throughout all consecutive T-L planes over time, approximating all tasks' ideal paths.

For convenience, we define the *local laxity* of a task T_i.

Definition 2 (Local Laxity) *The local laxity of the task T_i at the current time t_{cur} is denoted as $t_f - t_{cur} - l_i(t_{cur})$, where $1 \leq i \leq N$.*

The diagonal of the T-L plane has an important meaning: when a token reaches that side, it indicates that the task does not have any local laxity. Thus, if it is not selected immediately, then it cannot satisfy the scheduling objective of local feasibility. We call the diagonal of the T-L plane the *no local laxity diagonal* (or NLLD). All tokens are supposed to stay on or under the NLLD. We call the tokens both on NLLD and between NLLD and the bottom line of the T-L plane *active*. Active tokens have non-zero local remaining execution time. We call the tokens lying on or below the bottom line of the T-L plane *inactive*.

We observe that there are two time instants when the scheduling decision has to be made again on the T-L plane. One instant is when the local remaining execution time of a task is completely consumed, and it would be better for the system to run another task instead. When this occurs, the token reaches the horizontal line, as T_N does in Figure 3. We call this the *bottom reaching event* (or event-B). The other instant is when the local laxity of a task becomes zero so that the task must be selected immediately. When this occurs, the token reaches the NLLD, as T_1 does in Figure 3. We call this the *ceiling reaching event* (or event-C). To distinguish these events from traditional scheduling events, we call events B and C *secondary events*.

To design a scheduling algorithm for a single T-L plane, we propose *guidelines for locally-feasible scheduling (GLS)*, which are sufficient to provide local feasibility. (The sufficiency of the GLS is proved in Section 3.)

(GLS.1) At every scheduling event, as many active tokens as the number of processors allows (up to M) should be selected.

(GLS.2) At event-C, the token that invokes the event should be selected immediately.

Since an active token changes to inactive when it reaches the bottom line of the T-L plane, (GLS.1) implies following (GLS.3).

(GLS.3) At event-B, the token that invokes the event should not be selected if there are still at least M active tokens.

The fact that obeying these GLS's is sufficient for a scheduling policy to be able to provide local feasibility in a T-L plane allows significant flexibility in designing a scheduling algorithm. For example, a scheduling algorithm may assign task priorities for a particular application while ensuring local feasibility by following the GLS's. The priority assignment also could be either work-conserving or non-work-conserving, specify processor affinity, etc. At time t_f^k of TL^k, the time instant for the next task period event, the next T-L plane TL^{k+1} starts to be drawn and the designed scheduling algorithm remains valid. Thus, the scheduling algorithm is consistently applied for every event.

We consider *local scheduling algorithms (LSA)* designed following the GLS's. In the next section, we analytically prove that an LSA provides local feasibility in a T-L plane.

3 Properties of LSAs

A fundamental property of an LSA is its local feasibility—i.e., an LSA can consume all task local remaining execution times before the T-L plane ends. In this section, we prove that an LSA guarantees local feasibility on the T-L plane. We first suppose the case that $N > M$. The case that $N \leq M$ is considered later in Section 3.4.

3.1 Critical Moment

Figure 4 shows an example of token flows on a T-L plane. All tokens flow from left to right over time. The scheduling algorithm selects up to M tokens from at most N active tokens and they flow diagonally down. The others which are not selected take horizontal paths. When the event-C or B happens, denoted by t_j where $0 < j < f$, the LSA is invoked to make a scheduling decision.

Definition 3 (Local Utilization) *The local utilization $r_i(t_j)$ for a task T_i at time t_j is defined to be $\frac{l_i(t_j)}{t_f - t_j}$, which describes how much processor capacity needs to be utilized for executing T_i within the remaining time until t_f. Here, $l_i(t_j)$ is the local remaining execution time of task T_i at time t_j.*

Note that when k is omitted, it implicitly means the current k^{th} T-L plane.

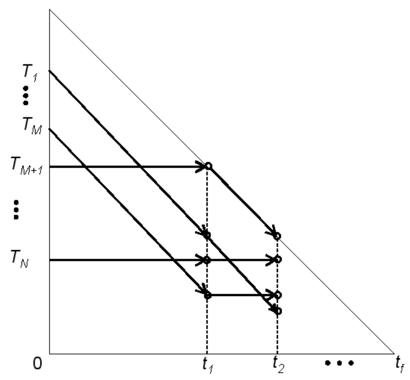


Fig. 4. Example of Token Flow

Theorem 4 (Initial Local Utilization Value in T-L plane) *Let all tokens arrive at the bottom line before t_f^{k-1} on the $(k-1)^{th}$ T-L plane. Then, the initial local utilization value $r_i(0) \leq u_i$ for all task T_i on the k^{th} T-L plane.*

PROOF. If all tokens arrive at t_f^{k-1} with $l_i(t_f^{k-1}) \leq 0$, then they can restart on the next T-L plane (the k^{th} T-L plane) from or below the positions where their ideal fluid schedule lines start. The slope of the fluid schedule path of task T_i is u_i. Thus, $r_i(0) = l_i(0)/t_f \leq u_i$. □

Well-controlled tokens have both departing and arriving points which are the same as, or lower than, those of their fluid schedule lines on the T-L plane (even though their actual paths on the T-L plane are different from their fluid schedule paths). Well controlled tokens are locally feasible. Note that we assume $u_i \leq 1$ and $\sum u_i \leq M$.

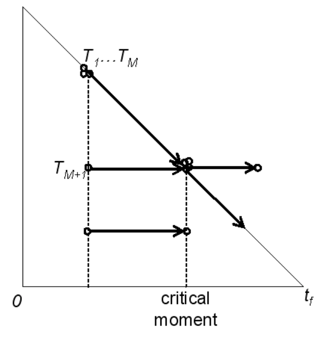


Fig. 5. Critical Moment

We define *critical moment* to be the sufficient and necessary condition that all tokens are not locally feasible. (Local infeasibility of the tokens implies that all tokens do not arrive at the bottom line of the T-L plane before t_f.) A critical moment is the first secondary event time when more than M tokens simultaneously reach the NLLD. Figure 5 shows this. Right after the critical moment, only M tokens from those on the NLLD are selected. The non-selected ones move out of the triangle, and as a result, they will not arrive at the bottom line of the T-L plane before t_f. Note that only horizontal and diagonal moves are permitted for tokens on the T-L plane.

Theorem 5 (Critical Moment) *At least one critical moment occurs if and only if tokens are not locally feasible on the T-L plane.*

PROOF. We prove both the necessary and sufficient conditions.

Only-if part. Assume that a critical moment occurs. Then, non-selected tokens move off of the T-L plane. Since only -1 and 0 are allowed for the slope of the token paths, it is impossible that the tokens off of the T-L plane reach the bottom line of the T-L plane.

If part. We assume that when tokens are not locally feasible, no critical moment occurs. If there is no critical moment, then the number of tokens on the NLLD never exceeds M. Thus, all tokens on the diagonal can be selected by the LSA up to time t_f. This contradicts our assumption that tokens are not locally feasible. □

We define *total local utilization* at the j^{th} secondary event.

Definition 6 (Total Local Utilization) *The total local utilization at the j^{th} secondary event, $S(t_j)$, is defined as $\sum_{i=1}^{N} r_i(t_j)$.*

Corollary 7 (Total Local Utilization at Critical Moment) *Supposing that the critical moment occurs at the j^{th} secondary event on a T-L plane, the total local utilization at the critical moment $S(t_j)$ is greater than M.*

PROOF. The local remaining execution time $l_i(t_j)$ for the tasks on the NLLD at the critical moment (at the j^{th} secondary event) is $t_f - t_j$, because the T-L plane is an isosceles triangle. Therefore, $S(t_j) = \sum_{i=1}^{N} r_i(t_j) = \sum_{i=1}^{M} \frac{t_f - t_j}{t_f - t_j} + \sum_{i=M+1}^{N} \frac{l_i(t_j)}{t_f - t_j} = M + \sum_{i=M+1}^{N} \frac{l_i(t_j)}{t_f - t_j} > M$. □

From the task perspective, the critical moment is the time when more than M tasks have no local laxity. Thus, the scheduler cannot make them locally feasible with M processors.

3.2 Event-C

Event-C happens when a non-selected token reaches the NLLD. Note that selected tokens never reach the NLLD. Event-C indicates that the task has no local laxity and hence should be selected immediately. Figure 6 illustrates this, where event-C happens at time t_c when the token T_{M+1} reaches the NLLD.

Note that this is under the basic assumption that there are more than M tasks, i.e., $N > M$. This implicit assumption holds in Section 3.2 and 3.3.

For convenience, we suppose that an LSA selects M tasks from T_1 to T_M and their tokens move diagonally. We give indices i to tokens inversely according to their local utilization—i.e., $r_i(t_j) \geq r_{i+1}(t_j)$ where $1 \leq i < M$ and $\forall j$, as shown in Figure 6. It also implies that for all $M+1 \leq i < N$ and j, $r_i(t_j) \geq r_{i+1}(t_j)$.

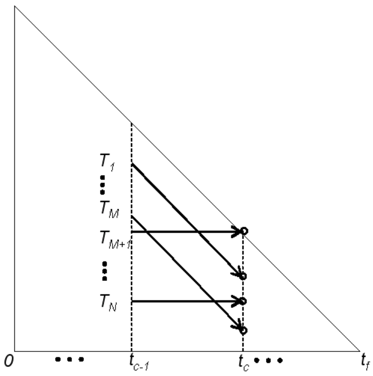


Fig. 6. Event-C

Lemma 8 (Sufficient and Necessary Condition for Event-C) *The event-C occurs at time t_c, if and only if $1 - r_{M+1}(t_{c-1}) \leq r_M(t_{c-1})$, where $r_i(t_{c-1}) \geq r_{i+1}(t_{c-1})$, $(1 \leq i < M)$ or $(M+1 \leq i < N)$.*

PROOF. The time when T_{M+1} reaches the NLLD is $t_{c-1} + (t_f - t_{c-1} - l_{M+1}(t_{c-1}))$. The time when the token T_M reaches the bottom of the T-L plane is $t_{c-1} + l_M(t_{c-1})$. We prove both the sufficient and necessary conditions. *If part.* We assume $1 - r_{M+1}(t_{c-1}) \leq r_M(t_{c-1})$.

$$1 - \frac{l_{M+1}(t_{c-1})}{t_f - t_{c-1}} \leq \frac{l_M(t_{c-1})}{t_f - t_{c-1}}$$

$$t_f - t_{c-1} - l_{M+1}(t_{c-1}) \leq l_M(t_{c-1}).$$

When adding t_{c-1} to both sides, we confirm that the time when T_{M+1} reaches the NLLD is prior or equal to the time when T_M reaches the bottom of the

T-L plane. Thus, event-C then occurs.
Only-if part. If the secondary event at time t_c is event-C, then token T_{M+1} must reach the NLLD earlier than when token T_M reaches the bottom of the T-L plane.

$$t_{c-1} + (t_f - t_{c-1} - l_{M+1}(t_{c-1})) \leq t_{c-1} + l_M(t_{c-1})$$

$$1 - \frac{l_{M+1}(t_{c-1})}{t_f - t_{c-1}} \leq \frac{l_M(t_{c-1})}{t_f - t_{c-1}}.$$

Thus, $1 - r_{M+1}(t_{c-1}) \leq r_M(t_{c-1})$. □

Corollary 9 (Necessary Condition for Event-C) *Event-C occurs at time t_c only if $S(t_{c-1}) > M(1 - r_{M+1}(t_{c-1}))$, where $r_i(t_{c-1}) \geq r_{i+1}(t_{c-1})$, $(1 \leq i < M)$ or $(M + 1 \leq i < N)$.*

PROOF.

$$S(t_{c-1}) = \sum_{i=1}^{M} r_i(t_{c-1}) + \sum_{i=M+1}^{N} r_i(t_{c-1}) > \sum_{i=1}^{M} r_i(t_{c-1}) \geq M \cdot r_M(t_{c-1}).$$

Based on Lemma 8, $M \cdot r_M(t_{c-1}) \geq M \cdot (1 - r_{M+1}(t_{c-1}))$. □

Theorem 10 (Total Local Utilization for Event-C) *When event-C occurs at t_c and $S(t_{c-1}) \leq M$, then $S(t_c) \leq M$, $\forall c$ where $0 < c \leq f$.*

PROOF. We define $t_c - t_{c-1} = t_f - t_{c-1} - l_{M+1}(t_{c-1})$ as δ. Total local remaining execution time at t_{c-1} is $\sum_{i=1}^{N} l_i(t_{c-1}) = (t_f - t_{c-1})S(t_{c-1})$ and it decreases by $M \times \delta$ at t_c as M tokens move diagonally. Therefore,

$$(t_f - t_c)S(t_c) = (t_f - t_{c-1})S(t_{c-1}) - \delta M.$$

Note that when event-C occurs, M tokens move diagonally from t_{c-1} to t_c according to (GLS.1).

Since $l_{M+1}(t_{c-1}) = t_f - t_c$,

$$l_{M+1}(t_{c-1}) \times S(t_c) = (t_f - t_{c-1})S(t_{c-1}) - (t_f - t_{c-1} - l_{M+1}(t_{c-1}))M.$$

Thus,

$$S(t_c) = \frac{1}{r_{M+1}(t_{c-1})}S(t_{c-1}) + (1 - \frac{1}{r_{M+1}(t_{c-1})})M. \tag{1}$$

Equation 1 is a linear equation as shown in Figure 7.

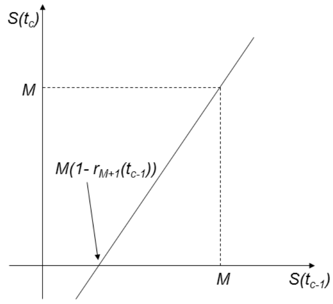


Fig. 7. Linear Equation for event-C

According to Corollary 9, when event-C occurs, $S(t_{c-1})$ is more than $M \cdot (1 - r_{M+1}(t_{c-1}))$. Since we also assume $S(t_{c-1}) \leq M$, we only consider the range from $M \cdot (1 - r_{M+1}(t_{c-1}))$ to M on the x-axis in Figure 7. Therefore, $S(t_c) \leq M$. □

3.3 Event-B

Event-B happens when a selected token reaches the bottom side of the T-L plane. Note that non-selected tokens never reach the bottom. Event-B indicates that the task has no local remaining execution time so it would be better to give the processor time to another task for execution.

Event-B is illustrated in Figure 8, where it happens at time t_b when the token of T_M reaches the bottom. As we do for the analysis of event-C, we suppose that an LSA selects M tasks from T_1 to T_M and we give indices i to tokens inversely according to their local utilization—i.e., $r_i(t_j) \geq r_{i+1}(t_j)$ where $1 \leq i < M$. It also implies that $r_i(t_j) \geq r_{i+1}(t_j)$ where $M + 1 \leq i < N$ and $\forall j$.

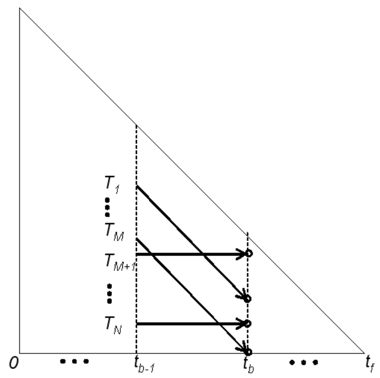


Fig. 8. Event-B

Lemma 11 (Sufficient and Necessary Condition for Event-B) *Event-B occurs at time t_b, if and only if $1 - r_{M+1}(t_{b-1}) \geq r_M(t_{b-1})$, where $r_i(t_{b-1}) \geq r_{i+1}(t_{b-1})$, ($1 \leq i < M$) or ($M \leq i < N$).*

PROOF. The time when T_M reaches the bottom and the time when T_{M+1} reaches the NLLD are respectively $t_{b-1} + l_M(t_{b-1})$ and $t_{b-1} + (t_f - t_{b-1} - l_{M+1}(t_{b-1}))$. We prove both sufficient and necessary conditions.
If part. We assume $r_M(t_{b-1}) \leq 1 - r_{M+1}(t_{b-1})$.

$$\frac{l_M(t_{b-1})}{t_f - t_{b-1}} \leq 1 - \frac{l_{M+1}(t_{b-1})}{t_f - t_{b-1}}$$

$$l_M(t_{b-1}) \leq t_f - t_{b-1} - l_{M+1}(t_{b-1}).$$

When adding t_{b-1} to both sides, we confirm that the time when T_M reaches the bottom is prior to the time when T_{M+1} reaches the NLLD, which is event-B.
Only-if part. If the secondary event at time t_b is event-B, then token T_M must reach the bottom earlier than when token T_{M+1} reaches the NLLD.

$$t_{b-1} + l_M(t_{b-1}) \leq t_{b-1} + (t_f - t_{b-1} - l_{M+1}(t_{b-1})).$$

$$\frac{l_M(t_{b-1})}{t_f - t_{b-1}} \leq 1 - \frac{l_{M+1}(t_{b-1})}{t_f - t_{b-1}}.$$

Thus, $r_M(t_{b-1}) \leq 1 - r_{M+1}(t_{b-1})$. □

Corollary 12 (Necessary Condition for Event-B) *Event-B occurs at time t_b only if $S(t_{b-1}) > M \cdot r_M(t_{b-1})$, where $r_i(t_{b-1}) \geq r_{i+1}(t_{b-1})$, $(1 \leq i < M)$ or $(M + 1 \leq i < N)$.*

PROOF.

$$S(t_{b-1}) = \sum_{i=1}^{M} r_i(t_{b-1}) + \sum_{i=M+1}^{N} r_i(t_{b-1}) > \sum_{i=1}^{M} r_i(t_{b-1}) \geq M \cdot r_M(t_{b-1}).$$

□

Theorem 13 (Total Local Utilization for Event-B) *When event-B occurs at time t_b and $S(t_{b-1}) \leq M$, then $S(t_b) \leq M$, $\forall b$ where $0 < b \leq f$.*

PROOF. $t_b - t_{b-1}$ is $l_M(t_{b-1})$. The total local remaining execution time at t_{b-1} is $\sum_{i=1}^{N} l_i(t_{b-1}) = (t_f - t_{b-1})S(t_{b-1})$, and this decreases by $M \cdot l_M(t_{b-1})$ at t_b as M tokens move diagonally. Therefore:

$$(t_f - t_b)S(t_b) = (t_f - t_{b-1})S(t_{b-1}) - M \cdot l_M(t_{b-1}).$$

Since $t_f - t_b = t_f - t_{b-1} - l_M(t_{b-1})$,

$$(t_f - t_{b-1} - l_M(t_{b-1}))S(t_b) = (t_f - t_{b-1})S(t_{b-1}) - M \cdot l_M(t_{b-1}).$$

Thus,

$$S(t_b) = \frac{1}{1 - r_M(t_{b-1})}S(t_{b-1}) - \frac{r_M(t_{b-1})}{1 - r_M(t_{b-1})}M. \tag{2}$$

Equation 2 is a linear equation as shown in Figure 9.

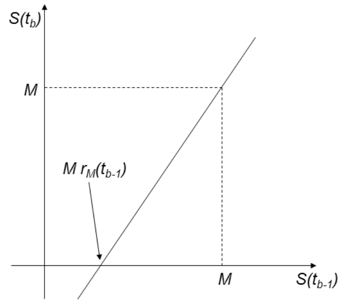


Fig. 9. Linear Equation for Event-B

According to Corollary 12, when event-B occurs, $S(t_{b-1})$ is more than $M \cdot r_M(t_{b-1})$. Since we also assume $S(t_{b-1}) \le M$, we only consider the range from $M \cdot r_M(t_{b-1})$ to M on the x-axis in Figure 9. Therefore, $S(t_b) \le M$. □

3.4 Local Feasibility

We now establish LSAs' local feasibility on the T-L plane based on our previous results.

In Section 3.3 and 3.2, we suppose that $N > M$. When less than or equal to M tokens only exist, they are always locally feasible by an LSA on the T-L plane.

Theorem 14 (Local Feasibility with Small Number of Tokens) *When $N \le M$, tokens are always locally feasible by an LSA.*

PROOF. We assume that if $N \le M$, then tokens are not locally feasible by an LSA. If there is not local feasibility, then there should exist at least one critical moment on the T-L plane by Theorem 5. Critical moment implies at least one non-selected token, which contradicts our assumption since all tokens are selectable according to (GLS.1). □

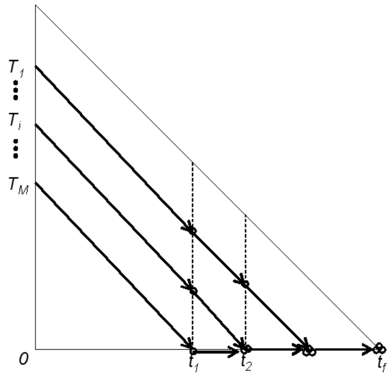


Fig. 10. Token Flow when $N \leq M$

Theorem 14 is illustrated in Figure 10. When the number of tasks is less than the number of processors, an LSA can select all tasks and execute them until their local remaining execution times become zero.

We also observe that at every event-B, the number of active tokens is decreasing. In addition, the number of event-B's in this case is at most N, since it cannot exceed the number of tokens. Another observation is that event-C never happens when $N \leq M$ since all tokens are selectable and move diagonally.

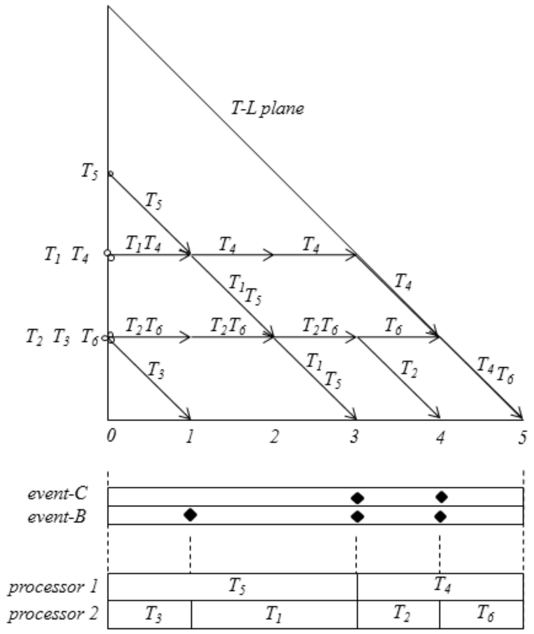
Now, we discuss the local feasibility when $N > M$.

Theorem 15 (Local Feasibility with Large Number of Tokens) *When $N > M$, tokens are locally feasible by an LSA when $S(t_0) \leq M$.*

PROOF. We prove this by induction, based on Theorems 10 and 13. Those theorems show that if $S(t_{j-1}) \leq M$, then $S(t_j) \leq M$, where j is the moment when secondary events occur. Since we assume that $S(t_0) \leq M$, $S(t_j)$ for all j never exceeds M at any secondary event including event-C's and event-B's. When $S(t_j)$ is at most M for all j, there should be no critical moment on the T-L plane, according to the contraposition of Corollary 7. By Theorem 5, there being no critical moment implies that they are locally feasible. □

When $N(> M)$ number of tokens are on the T-L plane and their $S(t_0)$ is at most M, event-C and B occur without any critical moment according to Theorem 15. Whenever event-B happens, the number of active tokens decreases until there are M remaining active tokens. Then, according to Theorem 14, all tokens are selectable so that they arrive at the bottom line of the T-L plane with consecutive event-B's.

Figure 11 shows one example of T-L plane-based schedules. Six tasks and two processors are assumed for the example, and the local remaining execution



Local remaining execution times for an example

task	l_i
T_1	2
T_2	1
T_3	1
T_4	2
T_5	3
T_6	1

Fig. 11. An Example Schedule by T-L Plane-based Approaches

time of each task within the time interval $(0, 5)$ is given by the table. Whenever a scheduling decision should be made, we assume that a T-L scheduling algorithm selects tasks to execute according to the GLS. If there is no task that ought to be selected by the GLS, the scheduler selects any task randomly. Note that the GLS is sufficient to provide local feasibility even with this restricted random selection. The T-L plane shows the flow of each token. The figure also shows when event-B and event-C occur over time as denoted by the small filled squares.

At time 0, we assume T_3 and T_5 are selected. At time 1, T_3 generates an event-B by reaching the bottom line and then, we assume T_1 is selected, based on (GLS.1). At time 3, T_1 and T_5 invoke event-B while T_4 invokes an event-C. According to (GLS.2), T_4 should be selected immediately. Simultaneously, one more task should be selected by (GLS.1). We assume T_2 is selected. At time 4, T_2 and T_6 generate an event-B and event-C respectively. Thus, T_6 should be immediately selected according to (GLS.2).

4 T-L Plane-based Scheduling Algorithm

Recall that T-L plane-based scheduling consists of two phases: LD-P where each task's local remaining execution time within a single T-L plane is determined and LS-P where all tasks' local remaining execution times within a single T-L plane are consumed.

As long as $S(t_0)$ is less than or equal to M as in Theorem 15, any LSA designed under the GLS's is applicable to LS-P. For LD-P, on the other hand, Figure 2 illustrates that the local remaining execution time at time $t_0(=0)$ for each task, $l_i(t_0)$, is determined from the fluid schedule line of each task. Thus, $l_i(t_0)$ is either an integer or a real number. However, due to the hardware characteristics of actual processors, $l_i(t_0)$ should be an integral multiple of the resolution of the highest precision timer, which makes LD-P more difficult. Besides, in LD-P, we should ensure that all $S(t_0)$ of continuous T-L planes are at most M while we determine $l_i(t_0)$ of each task to be an integer. Since $l_i(t_0)$ of each token is an integer and only diagonal and horizontal moves are permitted, all $l_i(t_j)$ (where $0 \leq j \leq f$) are also integers.

We present T-L plane-based scheduling algorithms that use a part of Zhu's algorithm in [7] for LD-P and use an LSA for LS-P. The design objective here is to make the computational complexity of the algorithms as low as possible. To do so, we suppose that the run-time (e.g., an operating system kernel) offers the scheduler the list of tasks that are in the ready state. In addition, the kernel notifies the scheduler of the task's identifier (or its pointer) when the event-C or B occurs. More details are provided in this section.

4.1 A Non-Work-Conserving Scheduling Algorithm

A T-L plane-based scheduling algorithm is shown in Algorithm 1. The key idea of the algorithm is to use two linked lists—e.g., ζ and Q_{run}. ζ is for tasks on the T-L plane and Q_{run} is a list of running tasks.

All tasks causing an event-C by reaching the NLLD are inserted (or pushed) into the front of Q_{run}, as in line 9 to 14, and the other running tasks are inserted (or pushed) into the back of Q_{run}. Thus, the tasks on NLLD are stacked from the front of Q_{run}, as illustrated in Figure 12. This ensures that at an event-C, the task at the back is not on NLLD and should be removed first, which simply takes $O(1)$ computational cost. Note that the number of tasks on the NLLD does not exceed M according to Theorem 15 and thus, the size of Q_{run} never exceeds M. The replaced task t_2 is inserted into the front of ζ as in line 13, since t_2 is still active.

The task causing an event-B turns into the inactive state, and it is removed from Q_{run} and inserted into the back of ζ. Then, if the task from the front of ζ is active (or has non-zero local remaining execution time), it is inserted into the front of Q_{run}. If the task from the front of ζ is inactive (or has zero local remaining execution time), it is inserted into the back of ζ.

At the period boundary of each task, all tasks' local remaining execution times, l_i, are determined by calling `decideLocalParameters()`, which is the

Algorithm 1: A Non-Work-Conserving Approach

Input : $\mathbf{T}$=$\{T_1,...,T_N\}$, tasks in ready state
Output : Array of dispatched tasks to processors
    M-# of processors
    ζ - Ready queue,
    Q_{run} - Queue for running tasks,
        - Maximal size of Q_{run} is M,
    l_i - Local remaining execution time of T_i,
    t_1, t_2 - Temporary variables for tasks

if *event-C* **then**
    t_1 = `getTaskOfThisEvent`();
    `erase`(t_1, ζ);————; remove a task t_1 from ζ
    t_2 = `pop-back`(Q_{run});————; take a task out from the back of Q_{run}
    `push-front`(t_2, ζ);————; push a task t_2 into the front of ζ
    `push-front`(t_1, Q_{run});————; push a task t_1 into the front of Q_{run}
else if *event-B* **then**
    t_1 = `getTaskOfThisEvent`();
    `erase`(t_1, Q_{run});————; remove a task t_1 from Q_{run}
    `push-back`(t_1, ζ);————; push a task t_1 into the back of ζ
    t_2 = `pop-front`(ζ);————; take a task out from the front of ζ
    if t_2 *is not NULL* **then**
        if t_2 *has* l_i *greater than zero* **then**
            `push-back`(t_2, Q_{run});——; push a task t_2 into the back of Q_{run}
        else
            `push-front`(t_2, ζ);————; push a task t_2 into the front of ζ
else if *period of each task* **then**
    `decideLocalParameters`(ζ); ——; set all l_i's
    `clear`(Q_{run});
    for i 1 **to** M **do**
        T_i = `pop-front`(ζ);————; take a task out from the front of ζ
        `push-back`(T_i, Q_{run});————; push a task T_i into the back of Q_{run}
return Q_{run};

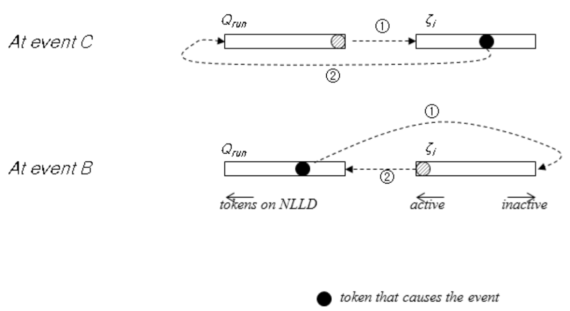


Fig. 12. Operations of Algorithm 1

LD-P step. As proved in Theorem 15, LD-P should determine l_i of all tasks to satisfy the inequality, $S(t_0) \leq M$. We adopt a part of Zhu's algorithm (or BF algorithm) in [7] for LD-P. The BF algorithm allocates some mandatory time units to each task, and one optional time unit to the eligible tasks at each boundary time (i.e., for each period of the task). Here, each task's l_i is defined as the sum of the mandatory time unit and the optional time unit by the BF algorithm. For an example, suppose there are six tasks and two processors. Each task is characterized by its execution time and deadline, $T_1 = (2, 5)$, $T_2 = (3, 15)$, $T_1 = (3, 15)$, $T_1 = (2, 6)$, $T_1 = (20, 30)$, and $T_1 = (6, 30)$. Their hyper-period, obtained as LCM (least common multiple), is 30. After time 0, the next task period occurs at 5. Based on Zhu's algorithm, each task's local remaining execution time from time 0 to 5 is computed as 2, 1, 1, 2, 3, and 1. On the first T-L plane, these local remaining execution times should be consumed. The next task period is at time 6. From time 5 to 6, each task's local remaining execution time is determined to be 1, 0, 0, 0, 1, and 0, all of which should be consumed during time 5 to 6. This is the same example in [7]. A more detailed description of the BF algorithm is found in [7]. After calling `decideLocalParameters()`, Q_{run} and ζ are initialized as shown in line 25 to 28.

In implementation, events C and B are invoked by timer expiration. At the period boundary of each task, algorithm 1 determines l_i of the ith task by calling `decideLocalParameters()`. As l_i is determined, the local laxity, $(t_f - l_i)$, can be easily obtained. Then the selected tasks start consuming l_i, and simultaneously the non-selected tasks start consuming their local laxities, $(t_c - l_i)$, where t_c is the current time. Thus, the subsequent scheduling event occurs at $min\{min_i\{l_i\}, min_j\{t_c - l_j\}\}$, where i is the index for selected tasks and j is the index for non-selected tasks. Every time the secondary event occurs, the next secondary event time can be determined in the same way after updating each task's l_i and $(t_c - l_i)$. The update computational complexity is $O(N)$. We assume that l_i and $(t_c - l_i)$, two of the states of tasks, are contained in data structures in the kernel, e.g., process control blocks in Real-time Linux, for the kernel to easily maintain.

Since a token remains on the bottom line after event-B though its remaining execution time (not its local remaining execution time) is not zero, the algorithm is a non-work conserving scheduling one, which implies that processors may be idle even when tasks are present in the ready queue and some processors are available. We consider a work-conserving scheduling algorithm in the following section.

The time complexity of this algorithm depends on the operations on Q_{run} and ζ as well as on the function `decideLocalParameters()`. If these are implemented as linked lists, the time complexities of all insert (`push`) and delete (`pop`) operations are $O(1)$. The operation `erase()` also costs $O(1)$ since we assume that the kernel notifies the scheduler of the task's identifier (or its pointer) when the event-C or B occurs. Thus, the time complexities of the algorithm are $O(1)$ when event-C or event-B occurs. At period boundaries of all tasks, on the other hand, the time complexity is determined by the time complexity of `decideLocalParameters()` and the for-loop in line 26. The time complexity of the BF algorithm is known to be $O(N)$ in [7] and therefore $O(max\{N, M\})$ is the time complexity of the entire algorithm.

4.2 A Work-Conserving Scheduling Algorithm

Given the fact that obeying the GLS's is sufficient for a scheduling algorithm to provide local feasibility in a T-L plane, algorithm 1 can be modified to be work-conserving as shown in Algorithm 2. Note that it is an example of the locally feasible scheduling algorithms. As long as it obeys GLS's, any priority assignment can be used.

Algorithm 2 uses three linked lists—e.g., ζ_k, ζ_{k+1}, and Q_{run}. ζ_k is for tasks on the current T-L plane and ζ_{k+1} is for tasks on the next T-L plane. ζ_k and ζ_{k+1} are switched at the period boundaries of all tasks. Q_{run} is a list for running tasks.

In Algorithm 2, we add e_i to represent the *actual* remaining execution time of T_i. e_i is compared with c_i that is updated after LD-P in line 26. c_i represents the remaining execution time after T_i consumes its allocated time by LD-P. The work-conserving policy that allows no processor idle time implies that e_i and c_i could be different. If e_i is greater than c_i, the token of T_i is active so the task should be inserted into the front of ζ^{k+1} as in line 32. Otherwise, it is inserted into the back of ζ^{k+1}. Therefore, after every task's period, all active tasks are collected at the front of ζ^k and all inactive tasks are collected at the back. Even after all tokens in ζ^k become inactive, a task can be moved into Q_{run} to run if some processors are available, as shown in line 23, which makes it work-conserving. The time complexity of Algorithm 2 is $O(max\{N, M\})$, the same as that of Algorithm 1. Table 1 is the summary of time complexities of the scheduling algorithms described in this paper.

4.3 Algorithm Overhead

One of the main concerns against global scheduling algorithms for multiprocessors is their overhead caused by frequent scheduler invocations. In [6], Srinivasan *et al.* identify three specific overheads:

(1) *Scheduling overhead*, which accounts for the time spent by the schedul-

Algorithm 2: A Work-Conserving Algorithm

Input : $\mathbf{T}$=$\{T_1,...,T_N\}$, tasks in ready state
Output : Array of dispatched tasks to processors
    M-# of processors
    ζ^k, ζ^{k+1} - Ready queues,
    Q_{run} - Queue for running tasks,
        - Maximal size of Q_{run} is M,
    l_i - Local remaining execution time of T_i,
    c_i - Remaining execution time of T_i,
    e_i - Actual remaining execution time of T_i,
    t_1, t_2 - Temporary variables for tasks

if *event-C* **then**
    t_1 = `getTaskOfThisEvent`();
    `erase`(t_1, ζ^k);————; remove a task t_1 from ζ^k
    t_2 = `pop-back`(Q_{run});————; take a task out from the back of Q_{run}
    `push-front`(t_2, ζ^k);————; push a task t_2 into the front of ζ^k
    `push-front`(t_1, Q_{run});————; push a task t_1 into the front of Q_{run}
else if *event-B* **then**
    t_1 = `getTaskOfThisEvent`();
    `erase`(t_1, Q_{run});————-; remove a task t_1 from Q_{run}
    `push-back`(t_1, ζ^k);————; push a task t_1 into the back of ζ^k
    t_2 = `pop-front`(ζ^k);————; take a task out from the front of ζ^k
    if t_2 *is not NULL* **then**
        `push-back`(t_2, Q_{run});————-; push a task t_2 into the back of Q_{run}
else if *periods of each task* **then**
    `attach`(Q_{run}, ζ^k); –; attach Q_{run} to ζ^k
    `decideLocalParameters`(ζ^k); ——; set all l_i's
    for i 1 **to** N **do** c_i=c_i-l_i;
    for i 1 **to** N **do** `update`(e_i);
    for i 1 **to** N **do**
        t_1 = `pop-front`(ζ^k);————; take a task out from the front of ζ^k
        if e_i *of* t_1 *is greater than its* c_i **then**
            `push-front`(t_1,ζ^{k+1});———; push a task t_1 into the front of ζ^{k+1}
        else `push-back`(t_1,ζ^{k+1});———; push a task t_1 into the back of ζ^{k+1}
    `switchReadyQueue`(ζ^k,ζ^{k+1});
    `clear`(Q_{run});
    for i 1 **to** M **do**
        T_i = `pop-front`(ζ^k);———; take a task out from the front of ζ^k
        `push-back`(T_i, Q_{run});———; push a task T_i into the back of Q_{run}
return Q_{run};

Table 1
Time Complexity

Scheduling algorithms	Time complexity
Zhu's algorithm [7]	$O(N)$
Non-work-conserving algorithm 1	$O(max\{N, M\})$
Work-conserving algorithm 2	$O(max\{N, M\})$

ing algorithm, including that for constructing schedules and ready-queue operations;

(2) *Context-switching overhead*, which accounts for the time spent in storing the preempted task's context and loading the selected task's context; and
(3) *Cache-related preemption delay*, which accounts for the time incurred in recovering from cache misses that a task may suffer when it resumes after a preemption.

Note that when a scheduler is invoked, the context-switching overhead and cache-related preemption delay may not happen. Srinivasan *et al.* also show that the number of task preemptions can be bounded by observing that when a task is scheduled (selected) consecutively for execution, it can be allowed to continue its execution on the same processor. This reduces the number of context-switches and possibility of cache misses. They bound the number of task preemptions under Pfair, illustrating how much a task's execution time inflates due to the aforementioned overhead. They show that for Pfair, the overhead depends on the time quantum size.

In contrast to Pfair, T-L plane-based scheduling is free from time quanta. We here use the number of scheduler invocations as a metric for overhead measurement, since it is the scheduler invocation that contributes to all three of the overheads previously discussed. We now derive an upper bound for the scheduler invocations under T-L plane-based scheduling.

Theorem 16 (Upper-bound on Number of Secondary Events in T-L plane) *When tokens are locally feasible on the T-L plane by Algorithm 1, the number of events on the plane is bounded within* $min\{N+1, t_f/t_{sys}\}$, *where* t_{sys} *is the system clock tick.*

PROOF. We consider two possible cases, when a token T_i reaches the NLLD, and when it reaches the bottom. After T_i reaches the NLLD, it will move along the diagonal to the rightmost vertex of the T-L plane, because we assume that the tasks are locally feasible. In this case, T_i raises one secondary event, event-C. Note that its final event-B at the rightmost vertex occurs together with the next event of another task's period (i.e., beginning of the new T-L plane). If T_i reaches the bottom, then the token becomes inactive and will arrive at the right vertex after a while. In this case, it raises one secondary event, event-B. Therefore, each T_i can cause one secondary event on the T-L plane. Thus, N number of tokens can cause $N+1$ number of events, which includes N secondary events and a task's period boundary at the rightmost vertex, when the system clock tick allows it. When $N+1$ is greater than t_f/t_{sys}, only t_f/t_{sys} number of secondary events can occur since the l_i's of all tokens are determined as integers by LD-P. □

The overhead of Algorithm 2 has a corresponding property except that early task completions should be considered in a work-conserving scheduling algorithm. However, it is still true that the number of secondary events linearly depends on N because the number of task completions in a T-L plane cannot exceed the number of N.

Theorem 17 (Upper bound of the Number of T-L Planes over Time) *When tasks can be feasibly scheduled by Algorithm 1, an upper bound of the number of the T-L planes in a time interval $[t_s, t_e]$ is:*

$$1 + \sum_{i=1}^{N} \left\lceil \frac{t_e - t_s}{p_i} \right\rceil,$$

where p_i is the period of T_i.

PROOF. Each T-L plane is constructed between two consecutive occurrences of task period boundaries, as shown in Section 2.2. The number of task period boundaries during the time between t_s and t_e is $\sum_{i=1}^{N} \left\lceil \frac{t_e - t_s}{p_i} \right\rceil$. The number of the T-L planes in a time interval is at most one more than the number of task period boundary occurrences. Thus, there can be at most $1 + \sum_{i=1}^{N} \left\lceil \frac{t_e - t_s}{p_i} \right\rceil$ number of T-L planes between t_s and t_e. □

Theorem 16 and 17 shows that the number of scheduler invocations of Algorithm 1 is primarily dependent on N and p_i—i.e., more tasks or shorter periods of tasks result in increased overhead. In other words, T-L plane-based approaches have overhead similar to that of many other traditional real-time scheduling algorithms. The overhead of EDF, for example, also increases as the number of tasks grows or the periods of tasks become shorter.

The number of scheduler invocations of the Pfair algorithm depends upon the time quantum size Q. In a time interval $[t_s, t_e]$, the number of Pfair scheduler invocations is $\lfloor \frac{t_e - t_s}{Q} \rfloor$. Thus, the overhead of Pfair does not depend on N or each p_i. When the time quantum becomes shorter, the overhead of Pfair increases as opposed to T-L plane-based approaches.

T-L plane-based scheduling consists of two phases, LD-P and LS-P. We propose LSA's for LS-P and adopt a part of Zhu's algorithm for LD-P. Zhu's algorithm utilizes McNaughton's rule for LS-P. Therefore, we first compare LSA's to McNaughton's rule for LS-P. Subsequently, we describe several properties of Zhu's algorithm for LD-P, which our algorithms partly inherit for LD-P.

McNaughton's rule that Zhu *et al.* uses for local scheduling was originally proposed to minimize the schedule length [14]. This scheduling problem is denoted as R$|pmtn|C_{max}$. The minimal schedule length D is given as $max\{\sum_{i=1}^{N} l_i(0)/M, max_i\{l_i(0)\}\}$ within a local schedule by McNaughton's rule, where N is the number of tasks and M is the number of processors. After ordering tasks arbitrarily, tasks are assigned to a processor to fill up to time D before tasks start being assigned to another processor.

Whereas McNaughton's rule provides the minimal schedule length, it may cause unnecessary migration for some cases. For example, when three processors and three tasks are given and each task's $l_i(0)$ is $\{3, 2, 2\}$ within a time interval [0, 3], D is calculated as 3. McNaughton's rule assigns T_1 to processor 1. Then, it assigns T_2 to processor 2 and assigns T_3 to processor 2 to fill up processor 2 to time 3. After filling up processor 2, it assigns the remaining part of T_3 to processor 3, which consequently causes a migration of T_3 from processor 2 to processor 3. In contrast, T-L plane-based local scheduling algorithms proposed in this paper do not cause this migration for this example since they would simply assign T_3 to processor 3 at the beginning. It is because T-L plane-based local scheduling is designed to make all tasks locally feasible. This scheduling problem is denoted as R$|pmtn|feasible$.

Above all, the GLS for LSA's gives significant flexibility to local scheduling design. For instance, when resource sharing constraints would be considered, the GLS would be a minimum requirement for a possible local scheduling algorithm to satisfy. Again, local scheduling algorithms for a periodic task set can be designed to follow the GLS, the sufficient condition for local feasibility.

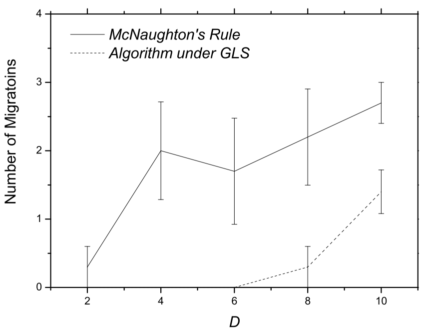
[7] characterizes the BF algorithm against the Pfair algorithm with several experimental results. Experimentally, BF has fewer scheduling events than Pfair does, especially when the maximum period of tasks becomes longer or the number of tasks becomes smaller. In addition, BF uses less processing time to generate the whole schedule than Pfair when the number of tasks is less than 100 because of fewer events. Our algorithms use BF for LD-P and the comparison against Pfair does not deviate much from [7]. Thus, in the following section, we focus on evaluating our algorithms against BF's local schedule experimentally, rather than comparing with Pfair.

5 Experimental Evaluation

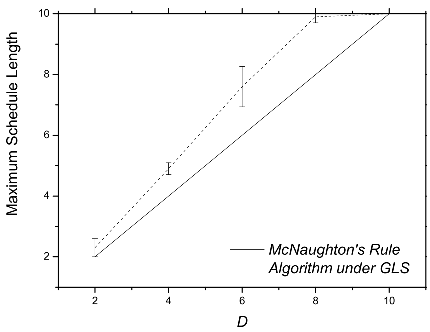
We conducted simulation-based experimental studies to compare T-L plane-based scheduling algorithms to some existing algorithms, and to validate our analytical results on overhead.

Comparison with McNaughton's rule [14]. Zhu's algorithm in [7] uses McNaughton's rule for local scheduling. We compare Algorithm 1 to McNaughton's rule for local scheduling. We first consider four processors and local scheduling within a time interval $[0, 10]$. We randomly generate tasks to be subject to D that is given as $\{2, 4, 6, 8, 10\}$. Several experiments are repeated and we show the results with a 95% confidence interval.

Figure 13(a) shows the number of migrations over varying D. Over all D, our algorithm generates fewer migrations than McNaughton's rule. However, we observe that McNaughton's rule always produces the minimum schedule lengths D, while our algorithm's schedule lengths are a little longer (but never exceed the time interval, 10) in Figure 13(b). This implies that McNaughton's rule may incur unnecessary migrations in order to minimize the schedule length, which is its original scheduling objective, as discussed in Section 4.4. We observe the similar results for the case of eight processors in Figure 14(a) and 14(b). However, the objective of our scheduling approach is different from McNaughton's—it is to complete all the tasks by their deadlines.



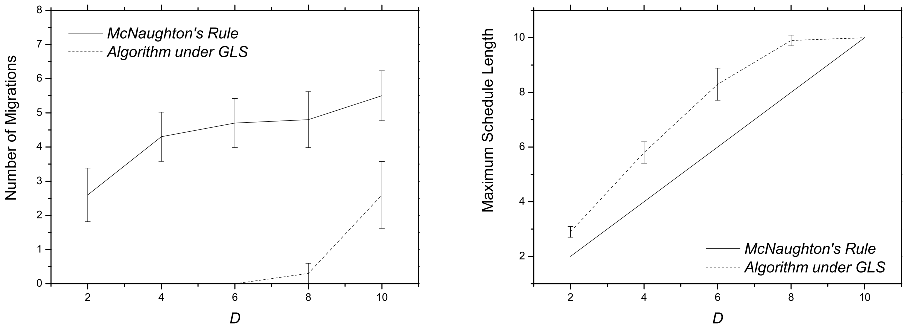
(a) Number of Migration



(b) Maximum Schedule Length

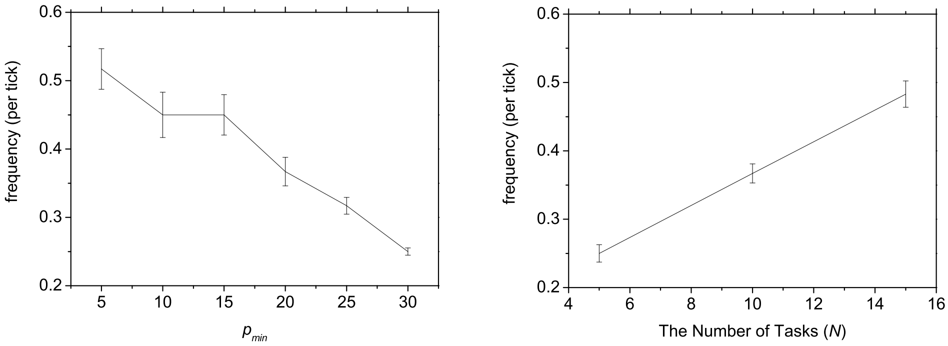
Fig. 13. Comparison with Four Processors

Overhead. To validate our analytical results on overhead, we considered an SMP machine with two processors, and several tasks running on the system. To evaluate the overhead in terms of the number of scheduler invocations, we define the scheduler invocation frequency as the number of scheduler invocations during a time interval $[t_s, t_e]$ divided by $[t_s, t_e]$. We set $[t_s, t_e]$ as 30 and we also set the system clock tick t_{sys} as 1. The total utilization is at most 2, the capacity of processors.



(a) Number of Migration (b) Maximum Schedule Length

Fig. 14. Comparison with Eight Processors



(a) (b)

Fig. 15. Scheduler Invocation Frequency

First, we randomly select task periods between 5 and 30 and the number of tasks around 5. Figure 15(a) shows that the actual frequency increases when the minimum task period p_{min} $(= \min\{p_1, ..., p_N\})$ becomes shorter. Second, we increase the number of tasks and Figure 15(b) shows that more running tasks leads to more scheduler invocations. In Figure 15, the error bar around each data point represents 95% confidence interval of that data point.

As Theorems 16 and 17 indicates, we observe that the smaller p_{min} and N proportionally affect the overhead. Thus, our experimental results validate our analytical results.

6 Conclusions

We introduce a novel abstraction for reasoning about timeliness of independent periodic task execution behavior on multiprocessors, called T-L plane. The abstraction allows viewing multiprocessor scheduling as scheduling on repeatedly occurring T-L planes, and correct scheduling on a single T-L plane leads to an optimal solution for all times. For a single T-L plane, we analytically show a sufficient condition, called GLS, to provide a locally feasible schedule. Based on these, we propose two examples of T-L plane-based real-time scheduling algorithms, which include non-work-conserving and work-conserving approaches. We also establish that the algorithm overhead is bounded in terms of the number of scheduler invocations, which is validated by our experimental (simulation) results. We experimentally show that our T-L plane-based local scheduling approach outperforms an existing approach in terms of the number of task migrations.

We believe that T-L plane-based algorithms are flexible to accommodate more relaxed task models having different aspects of our task model such as arrival pattern, time constraint, execution time, and dependency properties.

Since the scheduling in this paper is under an assumption that the next period time is known beforehand at the current period time, which allows a T-L plane to be established between those two adjacent events, it is not directly applicable to a sporadic task model, for example. For the sporadic task model, the T-L plane can be established between the current period time and the earliest possible time of the next period, and then the T-L plane scheduling algorithm can be applied within the time interval. After the earliest possible time of the next period, a different scheduling algorithm may be needed up to the time when the task actually arrives. Ascertaining the validity of that speculation, and making such a scheduling algorithm simple and efficient enough is an important future research direction.

7 Acknowledgements

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